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# ON SOME APPLICATIONS OF THE THEORY OF FREDHOLM TYPE LIMIT INTEGRAL EQUATIONS

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***Keywords:*** *Integral equations, Fredholm theory, almost periodic functions, Bohr spaces, boundary problems*

In this work we consider some problems of the theory of differential equations solutions of which could be found by application of the theory of limit Fredholm equations in Bohr spaces of almost periodic functions. In the paper one considers the analogs of some boundary problems and solve them by the theory of limit Fredholm equations. Since differential equations are not solvable in the space of almost periodic functions in general, we modify posing of the problems with the aim that the question was solvable in Bohr space.

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# FREDHOLM TİPLİ LİMİT İNTEQRAL TƏNLİKLƏR NƏZƏRİYYƏSİNİN BƏZİ TƏTBİQLƏRİ HAQQINDA

***Açar sözlər:*** *İnteqral tənliklər, Fredholm nəzəriyyəsi, sanki periodic funksiyalar, Bor fəzaları, sərhədd məsələləri*

Bu məqalədə biz diferensial tənliklər nəzəriyyəsinin bəzi elə məsələlərinə baxırıq ki, onları Bor mənada sanki periodik funksiyalar fəzasında limit inteqral tənliklər nəzəriyyəsinin köməyi ilə həll etmək mümkün olsun. Məqalədə bəzi sərhəd məsələlərinin analoqlarına baxılır və onlar limit inteqral tənliklər nəzəriyyəsinin köməyi ilə həll olunur. Ümumiyyətlə, diferensial tənliklərin sanki periodic funksiyalar fəzasında həlli olmaya bildiyindən biz məsələnin qoyuluşunu elə dəyişirik ki, onu Bor fəzasında həll etmək mümkün olsun.

***Н.Е.Аллахярова***

**О НЕКОТОРЫХ ПРИЛОЖЕНИЯХ ТЕОРИИ ПРЕДЕЛЬНЫХ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ ФРЕДГОЛЬМОВСКОГО ТИПА**

***Ключевые слова:*** *Интегральные уравнения, теория Фредгольма, почти-периодические функции, пространства Бора, граничные задачи*

В настоящей работе мы рассматриваем некоторые проблемы дифференциальных уравнений, решения которых могут быть получены применением теории предельных интегральных уравнений в пространствах почти периодических функций Бора. Рассматриваются аналоги некоторых граничных задач и решаются при помощи предельных интегральных уравнений Фредгольма. Так как, вообще говоря, дифференциальные уравнения не разрешимы в пространстве почти – периодических функций, мы так изменяем постановку задачи таким образом, чтобы ее можно было бы решать в пространстве Бора.

## 1. Introduction

There are many applications of the theory of integral equations of Fredholm type. In the literature applications of Fredholm theory to the questions of boundary problems of ordinary differential or equations with partial derivatives are best known. In this work we consider some problems of the theory of differential equations solutions of which could be found by application of the theory of limit Fredholm equations in Bohr spaces of almost periodic functions [3-6]. In the paper we consider the analogs of two boundary problems and solve them using the theory of limit Fredholm equations.

Since differential equations are not solvable in the space of almost periodic functions in general, we modify posing of the problems with the aim that the question was solvable in Bohr space. We state the equivalent variant of boundary problems in the space of almost periodic functions. The method of solution is based on the construction of the analog of Green function corresponding to the boundary problem and leading it to the solution of limit integral equations of Fredholm type.

Consider at first the boundary problem for ordinary differential equations of second order [2]. Let the functions $p$ and $q$ be real functions defined in the interval [0, 1]. Define linear operator as below:

$$L(u) = D_x(pD_xu) + qu,$$

where $D_x = \frac{d}{dx}$. Let we are given with boundary problem:

$$L(u) + \lambda u = 0, \quad (1)$$

with boundary conditions:

$$au(0) + bu'(0) = 0,$$
$$cu(1) + du'(1) = 0. \quad (2)$$

Considering the conditions (2), we suppose that the real numbers *a, b, c, d* satisfy the additional conditions

$$a^2 + b^2 \neq 0, c^2 + d^2 \neq 0.$$

We suppose also that

$$p \in C'[0,1], \qquad p \neq 0,$$
$$q \in C[0,1]. \quad (3)$$

Suppose that the number $\lambda = 0$ is not an eigenvalue for the operator $L(u)$. The problem is consisted in finding of all solutions *u* of the boundary problem (1)-(3), being not equally zero. In the theory of differential equations, some classical solutions of the considered problem are known. Let us briefly consider some results of this theory. It is interesting for us the method of investigation of this problem using integral equations. This method based on the construction of Green function corresponding to the given equation.

Consider the solution of the problem (1)-(3) by the method of Fredholm theory of II kind, briefly. In the condition on non-existence of eigenvalue 0 for the equation, the following lemma is true.

**Lemma 1.** For every pair of functions $u$ and *v* belonging to the class $C''$([0, 1]) we have

$$uL(v) - vL(u) = D_x\big(p(uv' - vu')\big).$$

Proof of this lemma is given in [6, p. 163].

As a consequence [6, p. 164], we obtain that if for the functions $u$ and *v* satisfying the conditions of Lemma 1 we have the equality

$$L(u) = L(v) = 0,$$

then

$$p(uv' - u'v) = C \neq 0,$$

**Lemma 2.** There exist functions $u$ and $v$ belonging to the class $C''([0,1])$ for which $L(u) = L(v) = 0$ and $au(0) + bu'(0) = 0$, $cu(1) + bu'(1) = 0$; these functions have such arbitrary multipliers that after their suitable choose the equality $p(uv' - u'v) = 1$ will be satisfied.

Proof of this lemma is given in [6, p. 165].

Consider briefly the method of construction of Green function.

**Theorem 1.** Let $y \in [0,1]$ be any point. For every boundary problem (1)-(3) there is only a unique function $K(x,y)$ of *x* satisfying the conditions:

1) The function $K(x,y)$ is continuous in [0, 1] as a function of *x;*

2) In every of sub-segments [0, y] and [y, 1] $K(x,y) \in C''$ and $L(K(x,y)) = 0$;
3) $aK(0,y) + bD_x(K(0,y))$=0, $cK(1,y) + dD_x(K(1,y))$=0;
4) $D_x(K(x,y-0)) - D_x(K(x,y+0)) = \frac{1}{p(y)}$.

Proof of this statement is given in the literature (see [6, p. 166]).

From Lemma 2, it follows an existence of a pair of functions $u$ and $v$ belonging to the class $C''([0,1])$ satisfying Lemma 2. Let's denote

$$K(x,y) = \begin{cases} v(y)u(x), 0 \le x \le y \\ u(y)v(x), y \le x \le 1. \end{cases}$$

This function is symmetric and is called to be Green function of considered boundary problem. The fundamental theorem proved by Hilbert states:

**Theorem 2.** The boundary problem (1)-(3) is equivalent to the following homogeneous Fredholm equation

$$u(x) = \lambda \int_0^1 u(y)\, K(x,y)dy.$$

The proof of the theorem is given in [6, p.171-172].

In [7, p. 251] analogical method of a construction of Green function used for equations of fourth order.

There are multidimensional problems which are also solved by using of a construction of multivariate Green function. For simplicity, we consider the case of three dimensions. Let us consider the equation

$$D_{xx}u + D_{yy}u + D_{zz}u + \frac{2a}{x}D_x u = 0 \ (0 < 2a < 1), x > 0.$$

Boundary problem demands that the solution of this equation must vanish on some surface (on the boundary of the domain). For the solution of this equation is developed the method of potentials. In some boundary problems for this equation it arises the question on defining of densities. This question leads to the solution of the Fredholm integral equation of the type:

$$\rho(s,t) = \lambda \iint_G K(x,y,s,t)\rho(x,y)dxdy,$$

with symmetric kernel.

In the case of limit integral equations, the question on applications of the theory is complica-ted due to properties of almost periodic functions. We will modify posing of the problem. Consider at first one-dimensional case.

## 2. On boundary problem in Bohr space of almost periodic functions

As it was noted in [2], differential equations are not solvable in the space of almost periodic functions, in general. By this reason, we modify posing of the problem in Bohr spaces. Consider at first the boundary problem (1)-(3). Using reasoning of the work [4], we shall state the equivalent variant of boundary problem in the space of almost periodic functions. For that, we need in consideration of ordinary integral equation in the segment [0, 1].

Construction of Green function corresponding to the boundary problem (1)-(3) is carried out by finding of two linearly independent solution of boundary problem, in [6]. Let us suppose that the Green function $K(x, y)$ is constructed. Then the general solution of the problem can be found by solving of the equation

$$u(x) = \lambda \int_0^1 u(y)\, K(x, y) dy. \qquad (4)$$

The solution defined in the segment [0, 1].

This solution produces a family of almost periodic functions as follows. Suppose that the function $K(x, y)$ belongs to the Lebesgue class $L_2([0, 1] \times [0, 1])$. Take the Fourier series of the function $K(x, y)$

$$K(x, y) \sim \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{n,m} e^{2\pi i(nx+my)}.$$

Now construction of almost periodic kernel can be performed by substituting of the variables $x$ and $y$ by the quantity $t\gamma$ with irrational number $\gamma$. In correspondence with the results of the work [4], substitute the equation (4) by limit integral equation

$$u(x) = \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x, \{s\gamma\}) u(\{s\gamma\}) ds. \qquad (*)$$

Since the function $u(\{s\gamma\})$ is an almost periodic, then the difference

$$\frac{\lambda}{T} \int_0^T K(x, \{s\gamma\}) u(\{s\gamma\}) ds - u(x)$$

is small uniformly with respect to $x = \{t\gamma\}$, as $T \to \infty$. Substituting $x$ by $\{t\gamma\}$ ({*} means the fractional part), we can find the sequence of values of *t*, being equal to $T_1,\ T_2, \dots,\ T_n \to \infty$ such that

$\{T_n\gamma\} \to 0$ . Then, in consent with boundary conditions (2), we have:

$$\lim_{n\to\infty} u(\{T_n\gamma\}) = 0.$$

The analogical relation we can write for the second boundary conditions, but with different sequen-ce of values $T'_1,\ T'_2$, ..., for which $\{T'_n\gamma\} \to 1$:

$$\lim_{n\to\infty} u(\{T'_n\gamma\}) = 0.$$

So, we can formulate the boundary problem in the space of Bohr almost periodic functions with boundary conditions:

$$\lim_{n\to\infty} (au(\{T_n\gamma\}) + b\gamma u'(\{T_n\gamma\})) = 0,$$
$$\lim_{n\to\infty} (au(\{T'_n\gamma\}) + b\gamma u'\big(\{T'_n\gamma\}\big)) = 0, \qquad (5)$$

and solve it by the help of Green's function.

Since $D_t\big(u(\{t\gamma\})\big) = D_x(u)D_t(t\gamma) = \gamma D_x(u)$, then $D_t(pD_t(u)) = \gamma^2 D_x(pD_x(u))$. So,

$$0 = \gamma^2(L(u) + \lambda u) = \gamma^2 D_x(pD_x(u)) + q\gamma^2 u + \gamma^2\lambda u.$$

Now we formulate the boundary problem as follows:

$$D_t(pD_t(u)) + \gamma^2(q+\lambda)u = 0 \qquad (6)$$

with boundary conditions (5).

**Theorem 3.** Let $\gamma$ be an irrational number and the sequences $(T_n), (T'_n)$ are defined as above. Then the solution of the equation (6) with boundary conditions (5) can be found as a solution of limit integral equation (*) above.

**Proof**. Consider at first the ordinary problem (4). As it was shown in [6, p.173] the set of eigenvalues of problem is an enumerable set. Every eigenvalue has a rank being equal to 1. In consent with the theorem VI of [6, p.174] the solution of the equation (4) is possible represent as follows

$$u(x) = \textstyle\sum_\nu C_\nu\psi_\nu(x);$$

moreover, the series is convergent uniformly and absolutely.

In consent with general case in [4], substituting $x = \{\gamma t\}, t \in R$, we get almost periodic function

$$v(t) = \sum_\nu C_\nu\psi_\nu(\{\gamma t\}) \qquad (7)$$

where the series converges uniformly and absolutely. We state that the function $v(t)$ is a solution of boundary problem (6) with boundary conditions (5).

Recalling that

$$L(u) = D_x(pD_x u) + qu,$$

we may write

$$D_t(pD_t v) = D_t(p(\{\gamma t\})\gamma \sum_{\nu} C_\nu \psi'_\nu(\{\gamma t\}) = \gamma^2 D_x(pD_x(u))|_{x=\{\gamma t\}}.$$

Since the function $u(x)$ is a solution of the equation

$$L(u) + \lambda u = 0,$$

then we have

$$D_t(pD_t(v)) + \gamma^2 qv + \lambda\gamma^2 v = \gamma^2(D_x(pD_x(u)) + qu + \lambda u)|_{x=\{\gamma t\}} =$$
$$= \gamma^2(L(u) + \lambda u)|_{x=\{\gamma t\}} = 0$$

for all real $t$. So, the obtained function $v(t)$ is a solution of the equation (6).

Prove that the boundary conditions (5) are also satisfied. Since

$$\lim_{n\to\infty} (\{T_n\gamma\}) = 0,$$
$$\lim_{n\to\infty} (\{T'_n\gamma\}) = 1,$$

and the series (7), having continuous members, converges uniformly, then the equalities (6) follow from (5), by passing to the limit. The theorem 3 is completely proved.

## 3. Boundary problem in Bohr space of bivariate almost periodic functions

Consider now boundary problems in two dimensional spaces. In the work [7] it was considered the equation

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x}u_x = 0, 0 < 2\alpha < 1, x > 0. \qquad (8)$$

In that work it was constructed the theory of potentials. For applications to boundary problems, was got the integral equation of Fredholm type. Using this method, we as above, will introduce the boundary problem in Bohr space of almost periodic functions and will solve it using the theory of limit integral equations.

Let us consider in brief the method of the work [7]. Let Γ be Lyapunov surface in the half space $x > 0$ bounded by simple connected open domain X on the plane x=0 and the surface Γ . Denote by $\mathrm{x} = \mathrm{x}(\mathrm{s},\mathrm{t}), \mathrm{y} = \mathrm{y}(\mathrm{s},\mathrm{t}), \mathrm{z} = \mathrm{z}(\mathrm{s},\mathrm{t})$ the parametric equation of the surface and

$$(\mathrm{s},\mathrm{t}) \in \overline{\Phi}, \Phi = \{(\mathrm{s},\mathrm{t}) | 0 \le \mathrm{s} \le 1, 0 \le \mathrm{t} \le 1\}.$$

Denote by $\gamma$ the common boundary of domains X and Γ. Suppose that:

1) Functions $\mathrm{x} = \mathrm{x}(\mathrm{s},\mathrm{t}), \mathrm{y} = \mathrm{y}(\mathrm{s},\mathrm{t}), \mathrm{z} = \mathrm{z}(\mathrm{s},\mathrm{t})$ have continuous partial derivatives which does not vanish simultaneously;

2) when the points of $\Gamma$ tends to the $\gamma$, then the surfaces intersect under right angle.

Consider now boundary problem of Dirichlet. It is required to find in D the solution of the equation (8), being continuous in $\overline{D}$, and satisfying the boundary conditions:

$$u|_\Gamma = \varphi(s,t), (s,t) \in \overline{\Phi}; u(0,y,z) = \tau_1(y,z), (y,z) \in \bar{X}, \quad (9)$$

where $\varphi(s,t)$ and $\tau_1(y,z)$ are given functions for which $\varphi(s,t)|_\gamma = \tau_1(y,z)|_\gamma$. On X we take $(y,z) = (s,t)$.

The solution of this equation is searched as a potential, with unknown density:

$$w_2(x,y,z) = \iint_\Gamma \mu_2(\theta,\vartheta)\, B_\nu^\alpha[q_2(\xi,\eta,\varsigma;x,y,z)]d\theta d\vartheta,$$

where $B_\nu^\alpha[q_2(\xi,\eta,\varsigma;x,y,z)]$ (*v=1, 2* ) is a fundamental solution of the equation (8). Using fact that for the satisfaction of the boundary condition it is required the equality $w_2(x,y,z) = \varphi_2(s,t)$, $(x,y,z) \in \Gamma$, we arrive at the integral equation for unknown density

$$\mu_2(s,t) - 2\iint_\Gamma \mu_2(\theta,\vartheta)\, K_2(s,t,\theta,\vartheta)d\theta d\vartheta = -2\varphi_2(s,t), \quad (10)$$

in which

$$K_2(s,t,\theta,\vartheta) = B_\nu^\alpha\big[q_2\big(\xi(\theta,\vartheta),\eta(\theta,\vartheta),\varsigma(\theta,\vartheta);x(s,t),y(s,t),z(s,t)\big)\big].$$

Using the method was applied above, we can formulate the analog of Dirichlet boundary problem in Bohr spaces. We suffice with posing and scheme of solution of the problem. For this we take some pair of real (irrational) numbers $(\delta,\lambda)$ independent over the field of rational numbers. We put $(s,t) = (\{p\delta\},\{q\,\lambda\}), (p,q) \in R \times R$. To every continuous function *f (s,t)* from the Lebesgue class $L_2$ (0, 1), we put in correspondence almost periodic function *f* $(p\delta, q\,\lambda)$. The analog of integral equation (10) will be a limit integral equation

$$\mu_2(p\delta,q\,\lambda) - 2\lim_{T\to\infty}\frac{1}{T}\int_0^T \lim_{T\to\infty}\frac{1}{T}\int_0^T \mu_2(p\delta,q\,\lambda)\, K_2(p\delta,q\,\lambda,\theta\delta,\vartheta\lambda)\, d\theta d\vartheta =$$

$$= -2\varphi_2(p\delta,q\,\lambda). \quad (11)$$

Substituting found density into the equality

$$w_2(x,y,z) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \lim_{T\to\infty}\frac{1}{T}\int_0^T \mu_2(p\delta, q\,\lambda)\, B_\nu^\alpha[q_2(\xi,\eta,\varsigma; x,y,z)]d\theta d\vartheta,$$

in which we must substitute $x = x(s,t), y = y(s,t), z = z(s,t)$, $(s,t) = (\{p\delta\},\{q\,\lambda\})$, we get the potential $w_2(x,y,z)$, with the same substitutions which will be a solution of the equation

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x(\{p\delta\},\{q\,\lambda\})}u_x = 0$$

with boundary conditions (9), written out in the limit form

$$u|_\Gamma = \lim_{m\to\infty}\lim_{n\to\infty}\varphi(\{p_m\delta\},\{q_n\lambda\});$$
$$u(0,y,z) = \lim_{m\to\infty}\lim_{n\to\infty}\tau_1(x(\{p_m\delta\},\{q_n\lambda\}), y(\{p_m\delta\},\{q_n\lambda\}))$$

where $(p_m)$ and $(q_n)$ are some sequence of real numbers tending to $+\infty$.

Note that in the work [7], it was proven that 2 is not an eigenvalue of the equation (11). So, the unique solution of the equation (11) can be found by using of resolvent of the limit integral equation, applying the theorem 3.2 of the work [4].

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